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A CONNECTION BETWEEN p -ADIC BANACH SPACES AND LOCALLY CONVEX
COMPACTOIDS

by

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ABSTRACT. For a vector space E over a non-archimedean valued field K a correspondence $p \mapsto p^0$ is established between seminorms p on E and compactoids p^0 in E^* .

Examination of it yields the solution of two open problems (see §4 and §8) and a reformulation of Serre's renorming problem (see §2). As a by-product results on metrizable compactoids are obtained (see §6).

§0 THE CORRESPONDENCE $p \mapsto p^0$.

Throughout this note K is a non-archimedean valued field, complete with respect to the metric induced by the nontrivial valuation $|\cdot|$.

Let E be a K -vector space, let E^* be its algebraic dual. A (non-archimedean) seminorm p on E is polar ([3], Definition 3.1), if

$$p = \sup \{ |f| : f \in E^*, |f| \leq p \}$$

Let P_E be the set of all polar seminorms on E .

For each $p \in P_E$ we set

$$p^0 = \{ f \in E^* : |f| \leq p \}$$

Then p^0 is an absolutely convex, edged ([3], §1b) subset of E^* . It is easy to see that p^0 is a closed compactoid ([3], §1e) with respect to the topology $\sigma(E^*, E)$, hence complete.

Let C_E^* be the set of all closed absolutely convex, edged compactoids in E^* with respect to $\sigma(E^*, E)$.

PROPOSITION 0. The map $p \mapsto p^0$ is a bijection of P_E onto C_E^* . Its inverse assigns to every $A \in C_E^*$ the seminorm p given by

$$p(x) = \sup \{ |f(x)| : f \in A \} \quad (x \in E)$$

Proof. We shall prove surjectivity of $p \mapsto p^0$ leaving the (easy) rest of the proof to the reader. So, let $A \in C_E^*$; we shall prove that $A = p^0$ where $p(x) = \sup \{ |f(x)| : f \in A \}$.

Obviously, $A \subset p^0$. Now let $g \in E^* \setminus A$, we prove that $g \notin p^0$. The space E^* is of countable type hence strongly polar ([3], Theorem 4.4). So by [3], Theorem 4.7, there exists a $\theta \in (E^*, \sigma(E^*, E))'$ such that $|\theta| \leq 1$ on A , $|\theta(g)| > 1$. But, by [3], lemma 7.1, θ has the form $f \mapsto f(x)$ for some $x \in E$. Thus, $|f(x)| \leq 1$ for $f \in A$, $|g(x)| > 1$ i.e., $p(x) \leq 1$ and $|g(x)| > 1$ and it follows that $g \notin p^0$.

Remarks.

1. Let K be spherically (= maximally) complete. Then each nonarchimedean seminorm p on E for which $p(x) \in \overline{|K|}$ ($x \in E$) is polar ([3], Remark following 3.1).
2. Let τ be the locally convex topology on E induced by all nonarchimedean seminorms i.e., τ is the strongest among all locally convex topologies on E . It is not hard to see that (E, τ) is a complete polar ([3], Definition 3.5) space and that (E, τ) and $(E^*, \sigma(E^*, E))$ are each others strong dual spaces.

§1 NORMS p FOR WHICH p^0 IS c' -COMPACT

Recall that an absolutely convex subset A of a locally convex space F over K is c' -compact if for each neighbourhood U of 0 in F there exist $x_1, \dots, x_n \in A$ (rather than $x_1, \dots, x_n \in F$) such that $A \subset U + \text{co} \{x_1, \dots, x_n\}$. (Here co indicates the absolutely convex hull)

THEOREM 1.1. For a polar seminorm p on a K -vector space E the following are equivalent.

(α) $p(x) \in |K|$ for each $x \in E$. Each onedimensional subspace of E has a p -orthocomplement.

(β) p^0 is c' -compact.

Proof. (α) \Rightarrow (β). By [7], Theorem 3.2, it suffices to prove that for each $\phi \in (E^*, \sigma(E^*, E))'$

$$\max \{ |\phi(f)| : f \in p^0 \}$$

exists. Since ϕ is an evaluation map we therefore have to show that

$$\max \{ |f(x)| : f \in p^0 \}$$

exists for each $x \in E$. This is obviously true if $p(x) = 0$. So assume $p(x) > 0$. Since $p(x) \in |K|$ we may assume that $p(x) = 1$. For such x we must prove

$$\max \{ |f(x)| : f \in p^0 \} = 1$$

By (α), Kx has a p -orthocomplement H . The function

$$f : \lambda x + h \mapsto \lambda \quad (\lambda \in K, h \in H)$$

is in E^* . We have $|f(x)| = 1$. For $\lambda \in K$, $h \in H$

$$|f(\lambda x + h)| = |\lambda| = p(\lambda x) \leq \max(p(\lambda x), p(h)) = p(\lambda x + h)$$

so that $f \in p^0$.

(β) \Rightarrow (α). Let $x \in E$. The map $f \mapsto |f(x)|$ ($f \in E^*$)

is a continuous seminorm on $(E^*, \sigma(E^*, E))$. By c' -compactness its restriction to p^0 has a maximum so there exists a $g \in p^0$ with $|g(x)| = p(x)$

(It follows that $p(x) \in |K|$). We prove that $\text{Ker } g$ is a p -orthocomplement of Kx . In fact, for $z \in \text{Ker } g$ we have

$$p(x+z) \geq |g(x+z)| = |g(x)| = p(x)$$

Then also

$$p(x+z) \geq p(z)$$

completing the proof of Theorem 1.1.

Note. It is not hard to see that (α) of above is equivalent too.

(γ) For each $x \in E$ there exists an $f \in E^*$ with $|f(x)| = p(x)$ and $|f| \leq p$.

For spherically complete K we obtain a simpler form of Theorem 1.1.

COROLLARY 1.2. Let K be spherically complete, let p be a seminorm on
 E for which $p(x) \in \overline{|K|}$ for all $x \in E$.

Then the following are equivalent.

(α) $p(x) \in |K|$ for each $x \in E$.

(β) p^0 is c' -compact.

Proof. By [1], lemma 4.35, each onedimensional subspace has a p -ortho-complement.

§2 APPLICATION: A NEW LIGHT ON SERRE'S RENORMING PROBLEM.

Consider the following two statements (*) and (**).

(*) Let E be a K -vector space and let $|| \cdot ||$ be a norm on E . Then there exists a norm $|| \cdot ||'$ on E , equivalent to $|| \cdot ||$, such that $||x||' \in |K|$ for all $x \in E$.

(**) Let K be spherically complete and let A be a complete absolutely convex compactoid in a Hausdorff locally convex space over K . Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a c' -compact B such that $A \subset B \subset \lambda A$.

The question as to whether (*) is true or not is known as Serre's renorming problem. See [2] for more details. We are able to reformulate this problem in terms of compactoids:

PROPOSITION 2.1. The above statements (*) and (**) are equivalent.

Proof. Assume (*). To prove (**) we may assume that A is edged. By [8], Theorem 3, A , as a topological module over $B(0,1) := \{\lambda \in K : |\lambda| \leq 1\}$, is isomorphic to a bounded submodule of K^I for some set I . Let E be the algebraic direct sum $\bigoplus_{i \in I} K_i$ where $K_i = K$ for all $i \in I$.

Then $(E^*, \sigma(E^*, E))$ is in a natural way isomorphic to K^I with the product topology. So we may assume that $A = p^0$ where p is a seminorm on E .

By (*) there exists a seminorm q , equivalent to p , such that $q(x) \in |K|$ for all $x \in E$. By a suitable scalar multiplication we can arrange that, in addition, $p \leq q \leq |\lambda|p$ for some $\lambda \in K$, $|\lambda| > 1$. Then

$$p^0 \subset q^0 \subset \lambda p^0$$

and q^0 is c' -compact by Corollary 1.2. This proves (**). Now assume (**).

To prove (*) we may assume (see [2]), that K is spherically complete.

Let p be a norm on E . By (**) there is a c' -compact B and a $\lambda \in K$, $|\lambda| > 1$ with $p^0 \subset B \subset \lambda p^0$. Then $B = q^0$ for some seminorm q on E . We have

$$p \leq q \leq |\lambda|p$$

and $q(x) \in |K|$ for all $x \in E$ by Corollary 1.2.

Note. Serre's renorming problem is still unsettled as far as I know.

§3 NORMS p FOR WHICH p^0 IS A KREIN-MILMAN COMPACTOID.

Recall that an absolutely convex subset A of a locally convex space over K is a KM-compactoid if it is complete and if $A = \overline{\text{co}} X$ where X is compact. (Here $\overline{\text{co}} X$ is the closure of $\text{co } X$).

Before stating the theorem we first make some simple observations. Let

p be a norm on E . We say that a collection $(e_i)_{i \in I}$ in E is a

p -orthonormal base of E if for each $x \in E$ there exist a unique

$(\lambda_i)_{i \in I} \in K^I$ such that $\{i \in I, |\lambda_i| \geq \epsilon\}$ is finite for each $\epsilon > 0$ and

$$x = \sum_{i \in I} \lambda_i e_i$$

$$p(x) = \max_i |\lambda_i|$$

If (E, p) is complete this definition coincides with the usual one.

LEMMA 3.1. Let (E, p) be a normed space, let (\hat{E}, \hat{p}) be its completion.

Then (E, p) has a p -orthonormal base if and only if (\hat{E}, \hat{p}) has a

\hat{p} -orthonormal base.

Proof. It is not hard to see that each p -orthonormal base of (E, p) is also a \hat{p} -orthonormal base of (\hat{E}, \hat{p}) . Conversely, let $(e_i)_{i \in I}$ be a \hat{p} -orthonormal base of (\hat{E}, \hat{p}) . For each $i \in I$, choose an $f_i \in E$ with $\hat{p}(e_i - f_i) \leq \frac{1}{2}$.

By [1], Exercise 5.C, $(f_i)_{i \in I}$ is a \hat{p} -orthonormal base of (\hat{E}, \hat{p}) .

Clearly $(f_i)_{i \in I}$ is a p -orthonormal base of (E, p) .

THEOREM 3.2. For a polar norm p on a K -vector space E the following are equivalent.

(α) (E, p) has a p -orthonormal base

(β) p^0 is a KM-compactoid.

Proof. (α) \Rightarrow (β). Let $(e_i)_{i \in I}$ be a p -orthonormal base of (E, p) . The formula

$$\phi(f) = (f(e_i))_{i \in I}$$

defines a map $\phi : p^0 \rightarrow B(0, 1)^I$. Straightforward verifications show that ϕ is an isomorphism of topological $B(0, 1)$ -modules. From [8], Theorem 16 we obtain that $B(0, 1)^I$, hence p^0 , is a KM-compactoid.

(β) \Rightarrow (α). Suppose $p^0 = \overline{\text{co}} X$ where X is a compact subset of E^* .

Let $C(X \rightarrow K)$ be the Banach space of all continuous functions $X \rightarrow K$, with the supremum norm $\| \cdot \|_\infty$. Then $C(X \rightarrow K)$ has an orthonormal base. ([1], Theorem 5.22).

The formula

$$\phi(x)(f) = f(x) \quad (f \in X)$$

defines a K -linear map $\phi : E \rightarrow C(X \rightarrow K)$. From

$$\|\phi(x)\|_\infty = \max_{f \in X} |f(x)| = \sup_{f \in \text{co}X} |f(x)| = \sup_{f \in p} |\bar{f}(x)| = p(x)$$

we obtain that ϕ is an isometry $(E, p) \rightarrow (C(X \rightarrow K), \|\cdot\|_\infty)$.

By Gruson's Theorem ([1], 5.9) $\overline{\phi(E)}$ has an orthonormal base. Then so has $\phi(E)$ by Lemma 3.1 and has E .

§4 APPLICATION: A COMPLETE c' -COMPACT SET WHICH IS NOT A KM-COMPACTOID.

We shall give a negative answer to the Problem following Theorem 1.7 of [6].

PROPOSITION 4.1. Let K be spherically complete, let $|K| = [0, \infty)$.

Then there exist a locally convex space F over K and a complete c' -compact subset $A \subset F$ which is not a KM-compactoid.

Proof. Let $E := l^\infty$ and let $F := (l^\infty)^*$ (with the topology we agreed upon in §0). Let p be the standard norm on l^∞ , and set $A := p^0$. Since, trivially, $p(x) \in |K|$ for all $x \in l^\infty$, we have that p^0 is c' -compact (Corollary 1.2).

However, it is known ([1], Cor. 5.19) that l^∞ has no orthogonal base so that (Theorem 3.2) p^0 is not a KM-compactoid.

§5 NORMS p FOR WHICH p^0 IS METRIZABLE.

THEOREM 5.1. For a polar seminorm p on a K -vector space E the following are equivalent.

(α) (E, p) is of countable type ([3], Definition 4.3).

(β) p^0 is metrizable.

Proof. $(\alpha) \Rightarrow (\beta)$. There exist e_1, e_2, \dots in E with $p(e_i) \leq 1$ for each i such that the K -linear span of e_1, e_2, \dots is p -dense in E . The formula

$$\phi(f) = (f(e_1), f(e_2), \dots)$$

defines a map $\phi : p^0 \rightarrow B(0,1)^{\mathbb{N}}$. Straightforward verifications show that ϕ is an isomorphism of topological $B(0,1)$ -modules of p^0 onto a submodule of $B(0,1)^{\mathbb{N}}$.

Now $B(0,1)^{\mathbb{N}}$ is metrizable (the product topology is induced by the metric $(a,b) \mapsto \sup_{i \in \mathbb{N}} |a_i - b_i| 2^{-i}$) hence so is p^0 .

$(\beta) \Rightarrow (\alpha)$. Let $\lambda \in K$, $|\lambda| > 1$. Since p^0 is a metrizable compactoid there exist, by [3], Proposition 8.2, $f_1, f_2, \dots \in \lambda p^0$ with $\lim_{n \rightarrow \infty} f_n = 0$ such that

$$p^0 \subset \overline{\text{co}} \{f_1, f_2, \dots\} \subset \lambda p^0$$

The map

$$\phi : x \mapsto (f_1(x), f_2(x), \dots) \quad (x \in E)$$

is K -linear, $\phi(E) \subset c_0$. We have for $x \in E$

$$||\phi(x)|| = \sup_{n \in \mathbb{N}} |f_n(x)| = \sup \{|g(x)| : g \in \overline{\text{co}} \{f_1, f_2, \dots\}\}$$

It follows that

$$p(x) \leq ||\phi(x)|| \leq |\lambda| p(x)$$

so that p is equivalent to $x \mapsto ||\phi(x)||$, a seminorm of countable type.

Hence, p is of countable type.

§6 APPLICATION: DESCRIPTION OF METRIZABLE COMPACTOIDS.

THEOREM 6.1. Let A be an absolutely convex subset of a Hausdorff locally convex space F over K. The following are equivalent.

- (α) A is a metrizable compactoid.
- (β) As a topological $B(0,1)$ -module, A is isomorphic to a submodule of $B(0,1)^{\mathbb{N}}$.
- (γ) As a topological $B(0,1)$ -module, A is isomorphic to a compactoid in c_0 .
- (δ) For each $\lambda \in K$, $|\lambda| > 1$ then exist $e_1, e_2, \dots \in \lambda A$ with $\lim_{n \rightarrow \infty} e_n = 0$ and $A \subset \overline{\text{co}} \{e_1, e_2, \dots\}$.
- (ε) There exist $e_1, e_2, \dots \in F$ with $\lim_{n \rightarrow \infty} e_n = 0$ and $A \subset \overline{\text{co}} \{e_1, e_2, \dots\}$.
- (η) There exists an ultrametrizable compact $X \subset F$ with $A \subset \overline{\text{co}} X$.

Proof. (α) \Rightarrow (β). It is not hard to see, by using the absolute convexity of A, that \overline{A} is also metrizable. As there is no harm in taking F complete we therefore may assume that A is complete. To prove (β) we also may assume that A is edged. By [8], Theorem 3, $A \subset B(0,1)^I \subset K^I$ for some set I. Like in the proof of Proposition 2.1 we may conclude that $A = p^0$ where p is a polar seminorm on $\bigoplus_{i \in I} K_i$ ($K_i = K$ for each i). Then p is of countable type by Theorem 5.1. From the proof of (α) \Rightarrow (β) of that Theorem we obtain an isomorphism $A = p^0 \xrightarrow{\sim} B(0,1)^{\mathbb{N}}$.

(β) \Rightarrow (γ). Choose $\lambda_1, \lambda_2, \dots \in K$, $|\lambda_1| > |\lambda_2| > \dots$, $\lim_{n \rightarrow \infty} \lambda_n = 0$. The formula

$$\phi((a_i)_{i \in \mathbb{N}}) = (\lambda_1 a_1, \lambda_2 a_2, \dots) \in c_0$$

defines a $B(0,1)$ -module isomorphism of $B(0,1)^{\mathbb{N}}$ onto $C := \overline{\text{co}} \{\lambda_1 e_1, \lambda_2 e_2, \dots\}$ where e_1, e_2, \dots are the standard unit vectors in c_0 . ϕ is a homeomorphism

$B(0,1)^{\mathbb{N}} \rightarrow C$, and maps A onto a compactoid in c_0 .

$(\gamma) \Rightarrow (\delta)$. See [3], Proposition 8.2.

$(\delta) \Rightarrow (\epsilon)$ is trivial.

$(\epsilon) \Rightarrow (\eta)$. $\{0, e_1, e_2, \dots\}$ is compact and ultrametrizable.

$(\eta) \Rightarrow (\alpha)$. We may assume that F is complete. It suffices to prove the metrizability of $B := \overline{\text{co}} X$.

B is a complete, edged compactoid. As before we may assume that $B = p^0$ for some polar seminorm p on some K -vector space E while $B \subset E^*$. The map $\phi : E \rightarrow C(X \rightarrow K)$ defined by

$$\phi(x)(f) = f(x) \quad (f \in X)$$

is an isometry $(E, p) \rightarrow (C(X \rightarrow K), || ||_{\infty})$.

Now X is ultrametrizable so by [1], Exercise 3.5, $C(X \rightarrow K)$ is of countable type. Hence so is p . By Theorem 5.1, $B = p^0$ is metrizable.

§7 NORMS p FOR WHICH $(p^0)^1$ IS OF FINITE TYPE.

Recall that an absolutely convex set A in a locally convex space F over K is of finite type if for each zero neighbourhood U in F there exists a finite-dimensional bounded set $S \subset A$ such that $A \subset U + S$.

Let us say that a seminorm p on a K -vector space E is of finite type if $\text{Ker } p = \{x \in E : p(x) = 0\}$ has finite codimension.

LEMMA 7.1. Let A be an absolutely convex subset of a locally convex space F whose topology is generated by a collection of seminorms of finite type. Then the following are equivalent.

(α) A is a compactoid of finite type.

(β) For each closed linear subspace H of finite codimension there is a finite dimensional bounded set $S \subset A$ with $A \subset H + S$.

Proof. $(\alpha) \Rightarrow (\beta)$. (Note. This implication holds for any locally convex space F .) We may assume $\llbracket A \rrbracket = F$.

H has the form $D^\perp := \{x \in F : f(x) = 0 \text{ for all } f \in D\}$ where D is a finite dimensional subspace of F' . Let f_1, \dots, f_n be a base of D . There exist $x_1, \dots, x_n \in F$ with $f_i(x_j) = \delta_{ij}$ ($i, j \in \{1, \dots, n\}$). Since $\llbracket A \rrbracket = F$ there exists a $\lambda \in K$, $\lambda \neq 0$ such that $\lambda x_i \in A$ for each $i \in \{1, \dots, n\}$. Set

$$U := \bigcap_{i=1}^n \{x \in F : |f_i(x)| \leq |\lambda|\}$$

Then U is a zero neighbourhood in F . A is a compactoid of finite type, so there exists a finite dimensional set $S_1 \subset A$ with $A \subset U + S_1$. Let $x \in U$. Write $x = y + z$ where

$$y := x - \sum_{i=1}^n f_i(x) x_i$$

$$z := \sum_{i=1}^n f_i(x) x_i$$

Now, since $x \in U$, $|f_i(x)| \leq |\lambda|$ for each i so that $z = \sum_{i=1}^n f_i(x) x_i \in A$.

Further, for each $j \in \{1, \dots, n\}$

$$f_j(y) = f_j(x) - \sum_{i=1}^n f_i(x) f_j(x_i) = f_j(x) - f_j(x) = 0$$

and it follows that $y \in D^\perp = H$. So $x = y + z$

$\in H + \llbracket x_1, \dots, x_n \rrbracket \cap A$. We see that

$$A \subset U + S_1 \subset H + S_2 + S_1$$

where $S_2 := \llbracket x_1, \dots, x_n \rrbracket \cap A$. Then (β) is proved with $S := S_1 + S_2$.

$(\beta) \Rightarrow (\alpha)$. Let U be a zero neighbourhood in F . Since continuous seminorms are of finite type, U contains a closed subspace H of finite codimension. By (β) there exists a finite dimensional set $S \subset A$ with S bounded and

$A \subset H + S$. Then $A \subset U + S$.

From now on we assume that the valuation on K is dense.

Recall that for an absolutely convex set B we have $B^{\perp} := \bigcup_{|\lambda| < 1} \lambda B$.

THEOREM 7.2. Let p be a polar norm on a K -vector space E . Then the following are equivalent.

- (α) For each finite dimensional subspace D of E there exists a seminorm q on E , q of finite type, $q \leq p$ and $q = p$ on D .
- (β) $(p^0)^{\perp}$ is of finite type.

Proof. (α) \Rightarrow (β). As each continuous seminorm on E^* is of finite type it suffices to prove, by Lemma 7.1, that for a closed subspace H of E^* of finite codimension there exists a finite dimensional set $S \in (p^0)^{\perp}$ such that $(p^0)^{\perp} \subset H + S$.

Now, by (α), there is a seminorm q of finite type, $q \leq p$ on E and $q = p$ on $D := H^{\perp}$. Let

$$S_1 := \{f \in E^* : |f| \leq q\}.$$

We see that S_1 is finite dimensional and since $q \leq p$ we have $S_1 \subset p^0$.

We now shall prove that $(p^0)^{\perp} \subset H + S$ where $S := (S_1)^{\perp}$.

In fact, let $f \in (p^0)^{\perp}$. Then there is a $\lambda \in K$, $0 < |\lambda| < 1$ with $|f| \leq |\lambda|p$.

Choose $\lambda' \in K$ with $|\lambda| < |\lambda'| < 1$.

We have $|f| \leq |\lambda| q$ on D (since $p = q$ on D) so we can extend f to a $g \in E^*$ with $|g| \leq |\lambda'| q$ on E . (This is because q is of finite type so that (E, q) is strongly polar.) Now write

$$f = f - g + g$$

Since $f = g$ on D we have $f - g \in D^{\perp} = H$.

Also, $|(\lambda')^{-1}g| \leq q$ so that $(\lambda')^{-1}g \in S_1$ i.e. $g \in (S_1)^{\perp} = S$.

(β) \Rightarrow (α). By lemma 7.1 there exists a finite dimensional set $S \subset (p^0)^{\perp}$ so that $(p^0)^{\perp} = D^{\perp} \cap (p^0)^{\perp} + S$.

Set $q(x) := \sup_{h \in S} |h(x)|$. ($x \in E$).

Then $q(x) = 0$ for all x in the space S^{\perp} which has finite codimension.

So q is of finite type.

Further, for $x \in E$ we have

$$q(x) = \sup_{h \in S} |h(x)| \leq \sup_{h \in (p^0)^{\perp}} |h(x)| = \sup_{h \in p^0} |h(x)| = p(x),$$

so $q \leq p$. Finally, if $x \in D$ then

$$p(x) = \sup_{f \in p^0} |f(x)| = \sup_{f \in (p^0)^{\perp}} |f(x)| = \sup_{\substack{h \in D^{\perp} \cap (p^0)^{\perp} \\ t \in S}} |h(x) + t(x)|$$

$$= \sup_{t \in S} |t(x)| = q(x). \text{ Hence, } p = q \text{ on } D.$$

§8 APPLICATION: A COMPLETE COMPACTOID IN c_0 THAT IS NOT OF FINITE TYPE.

If K is spherically complete each complete absolutely convex compactoid is of finite type (See [4], 2.3).

If K is not spherically complete the unit ball of c_0 is a complete compactoid for the weak topology but not of finite type (See [5], 1.6).

This is a non-metrizable compactoid. A compactoid in $(c_0, || ||)$, not of finite type, is given in [5], 1.4. However this compactoid is not closed. The following example provides an answer to the Problem following 1.5 in [5].

PROPOSITION 8.1. Let K be not spherically complete. Then there exists an absolutely convex complete compactoid in c_0 that is not of finite type.

Proof. Let $(K^V, | \cdot |)$ be the spherical completion of $(K, | \cdot |)$ in the sense of [1], Theorem 4.49. Let E be a K -subspace of K^V of countably infinite dimension and let p be the valuation $| \cdot |$ restricted to E . Then $x, y \in E$, $x \perp y$ in the sense of p implies $x = 0$ or $y = 0$. Obviously, the norm p is of countable type (hence polar) so, by Theorem 5.1, p^0 is metrizable and is by Theorem 6.1, isomorphic to a compactoid in c_0 . Suppose p^0 were of finite type. Then so would $(p^0)^\perp$ ([5], Proposition 2.4). By Theorem 7.2 we would have a seminorm q on E , $q \leq p$, q of finite type, $q(x) = p(x)$ for some $x \in E$, $x \neq 0$. But then $x \perp \text{Ker } q$ in the sense of p (If $q(y) = 0$ then $p(x-y) \geq q(x-y) = q(x) = p(x)$) which is impossible. So, p^0 is not of finite type.

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